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A Model of Subdiffusive Interface Dynamics with a Local Conservation of Minimum Height

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We define a new model of interface roughening in one dimension which has the property that the minimum of interface height is conserved locally during the evolution. This model corresponds to the limit $q \to \infty$ of the q-color dimer deposition-evaporation model introduced by us earlier [Hari Menon and Dhar, *J. Phys. A: Math. Gen.* **28**:6517 (1995)]. We present numerical evidence from Monte Carlo simulations and the exact diagonalization of the evolution operator on finite rings that growth of correlations in this model is subdiffusive with dynamical exponent $z \approx 2.5$. For periodic boundary conditions, the variation of the gap in the relaxation spectrum with system size appears to involve a logarithmic correction term. Some generalizations of the model are briefly discussed.

KEY WORDS: Interface growth; stochastic models; deposition-evaporation; conserved quantities; integrable models; Burgers equation; roughening; diffusion of polymers; Rouse model.

1. INTRODUCTION

The structure of fluctuating interfaces has been a major subject of study in recent years.[1,2] Most of these studies concern the large scale properties of a rough interface separating two phases. The recent spurt of interest in this subject started with the proposal of a nonlinear evolution equation for the interface height $h(\vec{x}, t)$ by Kardar, Parisi and Zhang (KPZ) one decade ago.[3] KPZ argued that consistent with the symmetries of the interface, the lowest order nonlinear term in the growth equation is proportional to

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$(\nabla h)^2$. Hence in a large number of physical situations the growing interface can be described by a noisy Burgers equation. In $1+1$ dimensions this nonlinear term is the most relevant perturbation, and it gives rise to a dynamical universality class characterized by the value of the dynamical exponent $z = 3/2$. Addition of higher order nonlinear terms, consistent with the symmetries, will not change the value of z.

Though this is indeed the case found in many physical situations, there are cases where the correlations of the interface are not described by KPZ exponents. For example, if there are some constraints present for the growth, we can expect the relaxation of the interface to become slower.[4, 5]

Some generalizations of the KPZ equation where the scalar height variable is replaced by an N-component vector,[6] or an $N \times N$ matrix[7] have been studied. But in both these cases it was found that the value of z remains unchanged in $1+1$ dimensions.

In this context, it is interesting to note that recently a class of deposition-evaporation models has been studied which in $1+1$ dimensions can be mapped to interface roughening models where the height variables are 2×2 matrices.[8–14] Monte Carlo simulations and numerical diagonalization studies have shown that these models do not belong to the KPZ dynamical universality class.[12, 13] In special cases, when the steady state shows long range spatial correlations, it was found that the fluctuations relax very slowly with $z \approx 5/2$. An example of this is the trimer deposition-evaporation (TDE) model.[8] Numerical diagonalization studies on small lattices shows that $z \approx 2.5$ for this model.[12] Another example is the q-color dimer deposition-evaporation (qDDE) model for $q > 2$.[13] Numerical studies suggests that this model is in the same universality class as the TDE model. (For the case of $q = 2$, qDDE model reduces to the exactly solved Heisenberg model, for which $z = 2$.[8]) Note that this value of z is quite different from the values z near 4 for the conserved KPZ equation discussed in refs. 4, 5.

A physically quite different, but mathematically related, problem where the same $z \sim 2.5$ exponent is encountered is the motion of ring polymers in a gel medium.[15] In this case Obukhov *et al.* have given heuristic scaling arguments that $z = 5/2$.[16] Some of these have been justified by Alon and Mukamel[17] using the fact that the probabilities of different configurations of the ring polymer in the steady state are exactly known. However, the result is still not rigorously established. Thus, it seems desirable to find a simpler model in this universality class that is better tractable analytically, and the dynamical behavior determined without using any not-obvious assumptions.

It is well known that many spin models become simpler, and in some cases even exactly solvable, in the limit when the number of components of the spin tends to infinity.[18] This has motivated us to the study of the

qDDE model in the limit of $q \to \infty$. Our main result is that in this limit, the qDDE model reduces to a simpler interface growth model with scalar heights. *This interface model has an interesting constraint that the minimum height of the interface is conserved locally during the growth.* This new model is somewhat simpler than the earlier models. The steady state of this model can be determined exactly however we have not succeeded in making much headway towards a full analytical solution so far. The model is interesting, and deserves further study for two reasons. Firstly, such a constraint seems to be of a qualitatively new kind, which has not been studied in the past. Since it affects the long-time behavior, it would be expected to have a continuum description. However, a local continuum evolution equation involving a finite number of derivatives which incorporates this constraint seems difficult to write down. Secondly, our numerical studies show the presence of logarithmic correction to the power-law scaling of the gap, which seems to be boundary condition dependent: it is present in the interface model, but not in the qDDE model. This unexpected feature also remains unexplained so far.

This paper is organized as follows: In Section 2 we define the interface model, and write down the stochastic matrix as the Hamiltonian of a quantum mechanical spin chain. We argue that conservation of minimum height locally in the model makes the relaxation slow. In Section 3 we calculate the average height of the interface in the steady state for a ring of size L, and show that this grows as $L^{1/2}$ for large L. In Section 4 we recapitulate the definition of the qDDE model and briefly list its known properties. In Section 5 we study the dynamics of the qDDE model in the limit of large q. We show that in this limit the model simplifies, and for time scales much greater than $1/q$ the model can be described by an effective Markovian dynamics. In Section 6 we establish an equivalence between this effective dynamics and the dynamics of the interface model defined in Section 2. In Section 7 we study this interface model using both Monte Carlo simulations and numerical diagonalization of the stochastic matrix for small lattice sizes. These studies show that this interface model is in the same universality class as the TDE model and the qDDE model with finite $q > 2$. In Section 8 we briefly discuss the case of asymmetric rates for $h \to h+2$ and $h \to h-2$. We also propose some higher dimensional generalizations of this interface model, which still conserve the minimum height of the interface locally.

2. DEFINITION OF THE INTERFACE MODEL

We consider the interface model on a one-dimensional lattice of size L. At any given time, the interface is specified by the integer height h_i at each

site i of the lattice. The heights are assumed to obey the restricted solid-on-solid (RSOS) condition

$$h_{i+1} - h_i = \pm 1 \tag{1}$$

for all i and all times. The interface evolves by the following Markovian local dynamics: at every site the interface height can be changed from h_i to either $h_i + 2$ or $h_i - 2$ with some rates, provided this will not violate the RSOS condition. *The rates for the transition $h_i \to h_i \pm 2$ depends on the next neighboring heights h_{i-2} and h_{i+2}.*

Equivalently, we can specify the interface in terms of slope variables $n_i = [h_{i+1} - h_i + 1]/2$, which takes only values 0 and 1. Then change in the height at site i corresponds to the exchange of the variables n at sites i and $i-1$. If we think of n's as occupation variables of a hard core lattice gas, this corresponds to the well-known exclusion process, with hopping rates between sites i and $i+1$ depends on the occupation at sites $i-1$ and $i+2$. In our model we assume that the rate for the rightward and leftward hoppings (or $h_i \to h_i + 2$ and $h_i \to h_i - 2$) are the same. Then there are 4 hopping rates, depending on the four possible states of the sites $i-1$ and $i+2$. Let us call these rates λ_1, λ_2, λ_3 and λ_4. These are shown in Fig. 1. At time $t = 0$, the interface height h_i is given to be 0 if i is odd, and 1 if i is even. At the boundaries, we can work with fixed boundary conditions corresponding to choosing the height to be 0 at $i = 0$ and $L + 1$ at all times. In numerical work we have used periodic boundary conditions so that $h_{L+1} = h_1$.

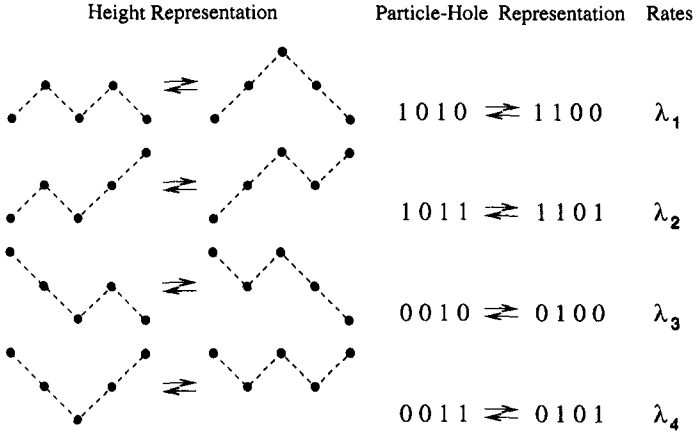


Fig. 1. Transition rates of the interface growth model.

When the hopping rate between sites i and $i+1$ is independent of n_{i-1} and n_{i+2} (simple exclusion process), the corresponding interface growth is described by the Hammersley–Edwards–Wilkinson equation,[22, 23] or equivalently, by the Rouse model of polymer dynamics.[19] This special case is exactly soluble, and the dynamical exponent z is known to be 2.

Even in the general case, with arbitrary $\{\lambda_i\}$, the detailed balance condition is satisfied and in the steady state all allowed configurations occur with equal weight. If all the rates λ_i, $i=1$ to 4 are nonzero, then it is easy to see that qualitative behavior of the relaxation is not changed much. Let $\Delta E(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the smallest nonzero eigenvalue of the relaxation matrix. The relaxation matrix is symmetrical, and can be thought of as the force matrix of a system of mass points connected by springs with spring-constants λ_i. As the eigenfrequencies are nondecreasing functions of the spring constants, it follows that

$$\frac{d}{d\lambda_i}\Delta E \geqslant 0 \tag{2}$$

It is then easy to deduce that

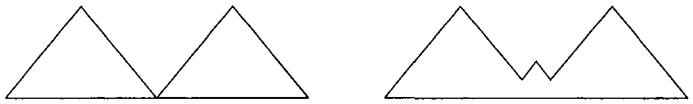
$$\lambda_{min}\,\Delta E(1,1,1,1) \leqslant \Delta E(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leqslant \lambda_{max}\,\Delta E(1,1,1,1) \tag{3}$$

where λ_{min} and λ_{max} are the smallest and largest values in a given $\{\lambda_i\}$.

Now, $\Delta E(1,1,1,1)$ is the lowest relaxation rate in the Rouse model, and it is known that for a ring of size L, it decreases as L^{-2} for large L. This implies that so long as λ_{min} and λ_{max} are finite, $\Delta E(\{\lambda_i\})$ decreases as L^{-2} for large L, and the dynamical exponent z remains 2. However, if λ_{min} becomes 0, then there is a possibility that we get a different universality class. *In this paper, we study the case when $\lambda_4 = 0$, and all other λ's are nonzero.* [The case $\lambda_1 = 0$, all other λ's nonzero is equivalent to this.] To keep the interface model transition rules left-right symmetric, we assume in addition that $\lambda_2 = \lambda_3$.

An important consequence of choosing $\lambda_4 = 0$ is that minimum of $\{h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}\}$ is conserved during a change of heights at i. This implies, in particular, that this interface dynamics conserves minimum of the full interface profile $\{h_i\}$. *The constraint that minimum height is locally conserved is a strong constraint that makes the dynamics slower than the Rouse dynamics.* Note that the conclusion $z \geqslant 2$ for our model follows immediately from the inequality (3).

As an illustration of this, consider the transition in Fig. 2 from the initial configuration I to final configuration F. Both these configurations are allowed, but to go from I to F, it takes a long process of restructuring.



I F

Fig. 2. An example of two configurations where it takes order of L^2 steps to reach one form the other.

The shortest route is to first completely erase one of the two "mountains", and then rebuild the final structure F from that. It is easy to convince oneself that other routes, essentially equivalent and requiring as many steps, are possible, but none requiring fewer can be found. For a pair of mountains of size L each, to go from I to F it takes order L^2 steps. In contrast in Rouse dynamics F can be reached from I in one step.

The constraint that minimum of height is locally conserved is approximately realized in some physical situations, which give rise to surface morphology sometimes known as "wedding cake" morphology with average width of surface increasing approximately linearly with the average height.[20] These are understood in terms of the existence of step-edge energy barriers of the Ehrlich–Schwoebel type, which inhibit jumps of atoms from higher steps to lower steps.[21] Clearly, this makes growth at height minima less likely. In the Monte Carlo simulations of Krug and Schimschack, the surface generated have deep ridges which seem to survive for a long time. Our model is simpler than theirs, but generates shapes of surfaces qualitatively similar. In particular, in both cases fluctuations in height are of the same order as mean height.

It is quite straightforward, but instructive, to write the stochastic matrix of the interface model as a quantum Hamiltonian $\hat{H}$. We consider a chain of L quantum mechanical spins $\{S_i\}$, $i=1$ to L. To each configuration $\{n_i\}$, we associate a spin configuration $\{S_i^z\}$ by the rule $n_i=(1+S_i^z)/2$. Then it is easily seen[24] that the quantum-mechanical Hamiltonian corresponding to our model is

$$\hat{H}=\sum_{i=1}^{L} f[\hat{S}_{i-1}^z, \hat{S}_{i+2}^z](\hat{\vec{S}}_i \hat{\vec{S}}_{i+1}-1) \tag{4}$$

where the function f takes arbitrary positive values except that $f(-1,+1)=0$. This is a four spin interaction Hamiltonian, which is not yet tractable analytically. Note that the Hamiltonian is not left-right symmetric though the original height model is. This is due to the fact the transformation from h_i to n_i breaks the reflection symmetry. It is invariant only under the simultaneous interchange $x \leftrightarrow -x$ and $n \leftrightarrow 1-n$.

3. STEADY-STATE PROPERTIES OF THE INTERFACE

Starting from an initial profile, the interface will grow until the system reaches a steady state. The average interface height in the steady state (the saturation height) can be computed exactly by mapping the interface profiles to paths of a random walk that is not allowed to cross the origin. Since the interface heights satisfy $h_{i+1} = h_i \pm 1$, the set $\{h_i\}$ forms the path of a random walk if we imagine h and i as space and time co-ordinates. The constraint that the walk is not allowed to cross the origin comes from the fact that the interface height at any point can not be negative.

Since the rules of the interface dynamics obey detailed balance, all accessible interface configurations have equal weight in the steady state. The average height of the interface then corresponds to the average displacement of the random walk from the origin, this can be computed exactly as follows: let us assume periodic boundary conditions and $L = 2n$, an even integer. We pick a point at random on the ring, and call it the origin. The number of configurations where height at the origin is h is given by the number of paths from $(i = 0, h)$ to $(i = 2n, h)$ which touches the line $h = 0$ but do not fall below. Let us denote this number by $\tilde{N}[(0, h); (2n, h)]$. If we denote $N_j[(0, h); (2n, h)]$ the number of paths from $(0, h)$ to $(2n, h)$ which do not fall below the line $h = j$,

$$\tilde{N}[(0, h); (2n, h)] = N_0[(0, h); (2n, h)] - N_1[(0, h); (2n, h)] \tag{5}$$

From the well known reflection principle,[25]

$$N_j[(0, h); (2n, h)] = M[(0, h); (2n, 2j - h - 2)] \tag{6}$$

where $M[(0, h); (2n, h')]$ denotes the number of paths from $(0, h)$ to $(2n, h')$ with no constraint on the path. Hence

$$\tilde{N}[(0, h); (2n, h)] = M[(0, h); (2n, -h)] - M[(0, h); (2n, -h - 2)] \tag{7}$$

Since

$$\begin{aligned} M[(0, h); (2n, h')] &= \binom{2n}{(|h' - h|/2) + n} \\ \tilde{N}[(0, h); (2n, h)] &= f(h) - f(h + 1) \end{aligned} \tag{8}$$

where,

$$f(h) = \binom{2n}{n + h} \tag{9}$$

Therefore the average height is given by

$$\langle h \rangle = \frac{\sum_{h=0}^{n} h[f(h) - f(h+1)]}{\sum_{h=0}^{n} [f(h) - f(h+1)]} \tag{10}$$

$$= \frac{1}{2}\left[2^{2n} \Big/ \binom{2n}{n} - 1 \right] \tag{11}$$

which increases as $\sqrt{\pi/8}\, L^{1/2}$ for large L. For an infinite system this means that starting from a profile which has the minimum possible heights ($\{h_i\} = \{0, 1, 0, 1, \ldots\}$), the average interface height will always grow. However due to the conservation of minimum height, it will always be tethered to $h = 0$ at least at one point. A similar situation occurs in the wetting problem, say in the Ising model below its critical temperature, when the boundary conditions at the two opposite ends of cylinder prefer two different values of magnetization and thus create a boundary between the predominantly up and predominantly down phases. If this domain boundary is near one of the ends of the cylinder in the beginning, it moves away from it with time under single-spin-flip dynamics. The driving force here, as in the problem we study, is due to higher entropy of the domain wall away from the surface, while the evolution rules are reversible and do not favor any particular phase.

4. A BRIEF REVIEW OF THE qDDE MODEL

The qDDE model is defined as follows: Consider a d-dimensional lattice, where at each site there is a discrete variable which can take q distinct discrete values (colors). The system undergoes a continuous time Markovian dynamics with the update rule that with rate 1 two neighboring spins having the same color, can simultaneously change their color to any of the other $(q-1)$ colors. For example consider the case of $q = 3$, and let a, b and c denote the 3 colors. An aa pair can become a bb pair or a cc pair with rate 1. In the same manner bb and cc pairs can change their color. The dynamical rule for this case can be stated as in Fig. 3.

The qDDE model has been studied in detail in $1+1$ dimensions.[13] The main feature of the model is that its phase space breaks up into an exponentially large number of dynamically disconnected sectors. The number of sectors scales as $(q-1)^L$ with system size. This strong nonergodic behavior is due to the presence of a conserved quantity in the model, called *irreducible string* (IS) which is defined as follows: A configuration of the qDDE model on a linear chain of length L can be represented by a string of L characters where the ith character represents the color of the ith spin.

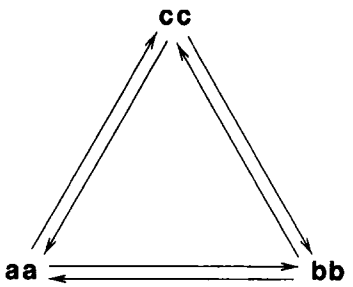


Fig. 3. Allowed transitions in the qDDE model for $q = 3$.

From this string delete all pairs of adjacent characters that are the same. Repeat this procedure on the resulting string until a string with no immediate repetition of characters is obtained. This string, whose length can not be reduced further by this reduction algorithm is the IS corresponding to given configuration. It can be shown that IS is conserved during the qDDE dynamics and each sector has a distinct IS.[13] As a consequence, IS can be used to uniquely specify a sector. The special sector for which the irreducible string is a null string (the original string is completely reducible) is called the *null sector* and it has long-range correlations in the steady state. The relaxation time of fluctuations in this sector also diverges with system size.

In any given sector, in the steady state all the configurations of that sector occur with equal probability. This follows from detailed balance. The number of configurations in each sector can be computed exactly, and it typically grows as $\exp(L)$. For example, in the null sector the number of configurations grows as $[4(q-1)]^{L/2}$.

In addition to IS, the qDDE model in one-dimension has another set of conserved quantities which corresponds to a symmetry called *recoloring symmetry* in the model. A qDDE configuration can be represented by the configuration of a polymer chain on a q-coordinated Bethe lattice. To see this, consider a q-coordinated Bethe lattice with its bonds colored with q different colors such that all the bonds meeting at any given site have different colors. We define the polymer chain corresponding to a given qDDE configuration as the L step walk which starts from the origin and follows the sequence of bonds such that their color is in the same sequence as the colors in the qDDE configuration from site 1 to site L. The chain may traverse a bond more than once. By definition, one end of the polymer is fixed at the origin. The conservation of the IS implies that the other end of the polymer chain is also fixed. Different positions of the second endpoint corresponds to different sectors. Thus there is a one-to-one correspondence between the L step polymer chain configurations on the Bethe

lattice and the configurations of the qDDE model. In this representation, the qDDE dynamics corresponds to a kind of reptation motion of the polymer chain on the Bethe lattice. A kink, consisting of two adjacent steps of the polymer chain where an immediate retraversal occurs, can jump to one of the neighboring sites on the Bethe lattice. The rules of the dynamics of the polymer chain are independent of the color of the bonds and hence recoloring of the bonds is a symmetry of the model. Note that this recoloring symmetry is not just the symmetry under permutations of colors which corresponds to a global rotation in the color space. It allows a local recoloring at each bond, and thus is more like a gauge symmetry.[3] Using the recoloring symmetry a large number of eigenstates of the stochastic matrix of the model can be computed exactly.[13] However these eigenstates are not the low lying eigenstates, which determine the long time behavior and hence the dynamical exponent z.

Note that the dynamics of the qDDE model can also be mapped to the dynamics of a polymer chain on a *q-coordinated regular lattice* provided that it is possible to color bonds of this lattice with q colors so that all the q bonds meeting at any one site have different colors. For example, for $q = 3$, we may choose the planar hexagonal lattice. As in the case of the Bethe lattice, a qDDE configuration maps uniquely to a configuration of a polymer chain. The null sector of the qDDE model with periodic boundary conditions map onto the dynamics of a ring polymer on this lattice. Note that this polymer ring is a "ghost ring" with no excluded volume interactions and two strands of polymers can go through each other so that entanglement effects are absent.

Time evolution in the qDDE model is diffusive in most sectors. The precise decay of the time-dependent correlation functions is different in different sectors, and depends on the IS of the sector. These different behaviors can be understood in terms of the hard-core random walkers with conserved spin (HCRWCS) model introduced in ref. 11. In this model, the positions of the IS characters are treated as non-crossing random walkers on a line. The corresponding dynamical exponent $z = 2$. The evolution is quite different in the null sector where there are no random walkers (more generally, in sectors where the length of the IS is a negligible fraction of L). In this sector the equal time spin-spin correlation function decays as a function of the distance r as $r^{-3/2}$. Thus the time-dependent spin-spin autocorrelation function would be expected to vary as $t^{-3/2z}$ for large t. Monte Carlo simulations and numerical diagonalization of the stochastic

[3] Of course, we cannot change the color only at a finite number of sites of the Bethe lattice without violating the constraint that all bonds meeting at a site have different colors.

matrix for small systems shows that the exponent $z \approx 2.5$. Thus, in this case, the growth of correlations in time is sub-diffusive.

5. DYNAMICS IN THE LARGE-q LIMIT

Consider the qDDE model on a chain of length L with open boundary conditions. Here we restrict to only the null sector but our following analysis is equally applicable to other sectors. In terms of polymer chain configurations, we consider the case where both the end points of the chain are fixed to be at the same site, i.e. at the origin. We divide the set of configurations in this sector into equivalence classes such that all configurations in a given equivalence class are related to each other by recoloring symmetry. As an example, in Table I all the equivalence classes are shown for $L = 6$ along with the number of configurations in each class. Each class has a unique topology of polymer chain configuration on the Bethe lattice. As an example the polymer chain configurations of equivalence classes $\mathbb{C}_1$, $\mathbb{C}_9$ and $\mathbb{C}_{10}$ are shown in Fig. 4.

Note that for $\mathbb{C}_8 - \mathbb{C}_{12}$, each bond of the Bethe lattice is traversed either twice or not at all by the polymer chain and these classes have more configurations than the other classes. In general if the polymer chain occupy n distinct bonds of the Bethe lattice, then the number of configurations in the corresponding equivalence class is approximately q^n for large q.

Table I. Equivalence Classes under Recoloring Symmetry of the qDDE Model in the Null Sector for $L = 6$[a]

Equivalence class	Representative element	Number of configurations
$\mathbb{C}_1$	*aaaaaa*	q
$\mathbb{C}_2$	*aaaabb*	$q(q-1)$
$\mathbb{C}_3$	*aaabba*	$q(q-1)$
$\mathbb{C}_4$	*aabbaa*	$q(q-1)$
$\mathbb{C}_5$	*abbaaa*	$q(q-1)$
$\mathbb{C}_6$	*bbaaaa*	$q(q-1)$
$\mathbb{C}_7$	*abbbba*	$q(q-1)$
$\mathbb{C}_8$	*aabbcc*	$q(q-1)(q-2)$
$\mathbb{C}_9$	*abbcca*	$q(q-1)(q-2)$
$\mathbb{C}_{10}$	*abbacc*	$q(q-1)(q-1)$
$\mathbb{C}_{11}$	*bbacca*	$q(q-1)(q-1)$
$\mathbb{C}_{12}$	*abccba*	$q(q-1)(q-1)$

[a] Here $a, b, c, \ldots$ represent different colors. Colors which are adjacent but represented by different symbols are assumed to be distinct. For example in $\mathbb{C}_{10}$, $a \neq b$ and $a \neq c$, but b and c need not be distinct.

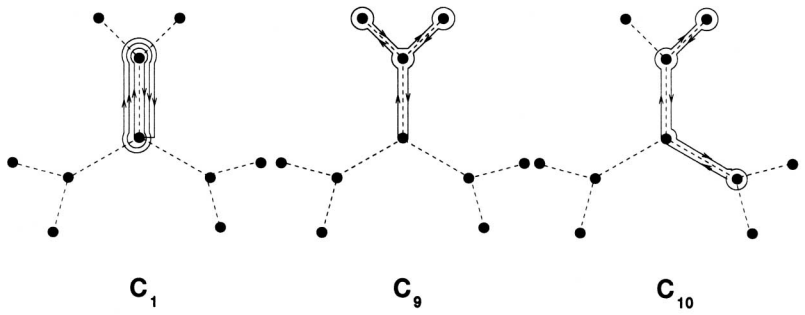


Fig. 4. Polymer chain configurations on the Bethe lattice corresponding to equivalence classes $\mathbb{C}_1$, $\mathbb{C}_9$ and $\mathbb{C}_{10}$ of the qDDE model on a chain of lenght $L=6$. For every site only 3 bonds are shown.

As in the steady state all configurations appear with equal probability, the weight of an equivalence class in the steady state is directly proportional to the number of configurations in it. This implies that in the $q \to \infty$ limit, only those equivalence classes where the polymer chain traverses each bond of the Bethe lattice exactly twice or not at all, will have a nonzero weight in the steady state. For example in the $L=6$ case, only the equivalence classes $\mathbb{C}_8 - \mathbb{C}_{12}$ will have a nonzero weight in the steady state in the $q \to \infty$ limit. If the state of the chain is examined at some instant in the steady state, with probability 1 it will belong to one of the equivalence classes $\mathbb{C}_8 - \mathbb{C}_{12}$. And the weight of all these equivalence classes are the same in this limit, which we can take as 1. We shall call such classes long-lived and the others short-lived.

In the steady state, a finite fraction of spins can flip at any instant. For the spin at site i to flip it is necessary that at least one of its neighbors (at site $i-1$ or $i+1$) should have the same color. The probability for this is given by $P = 2qN_{L-2}/N_L$, where N_L is the number of configurations in the null sector on a lattice of length L. As $N_L \sim [4(q-1)]^{L/2}$ for large L, $P = 1/2$ in the limit $q \to \infty$.

We have already seen that, in the large q limit, with probability 1 the equivalence class of a configuration in the steady state is long-lived. The average time the system spends in a particular configuration is very small, of order $1/qL$. This is because in a typical configuration there are order of L flipable pairs, and each such pair can go to any of the approximately q other states with rate 1. However most of these transitions are within the same equivalence class. For example, consider a local configuration of a flipable pair $|\cdots abbc\cdots\rangle$. If the pair bb changes its color to d which is different from both a and c, then the resulting configuration is equivalent to the old by recoloring symmetry. Short-lived classes have configuration of

the type $|\cdots aaac\cdots\rangle$, and within a time of order $1/q$ these will revert back to one of the type $|\cdots abbc\cdots\rangle$ or $|\cdots bbac\cdots\rangle$.

Thus, for large q, most of the dynamics of qDDE model involves transitions within a long-lived equivalence class. The only allowed inter-class transitions are from a long-lived to a short-lived equivalence class, and vice versa. These occur with rate Γ, where Γ is of order 1. Let us consider an allowed transition from equivalence class $\mathbb{C}$ to one $\mathbb{C}'$, where $\mathbb{C}$ is long-lived and $\mathbb{C}'$ is short-lived. $\mathbb{C}'$ being short-lived, in a time of order $1/q$ it makes a transition to a long-lived equivalence class $\mathbb{C}''$. Let us say it goes to long-lived classes $\mathbb{C}_1''$, $\mathbb{C}_2''$,... with probabilities $p_1, p_2,\ldots$. Therefore the effective transition rate from the long-lived class $\mathbb{C}$ to $\mathbb{C}_j$ is $p_j\Gamma$. In this way, in the limit of large q and for times $\gg 1/q$, we have a coarse-grained description of the stochastic evolution of the qDDE model as transitions between different long-lived equivalent classes, and this effective dynamics is *Markovian* with specified rates.

6. EQUIVALENCE TO THE INTERFACE MODEL

We now show that the effective dynamics of the qDDE model in the large q limit is equivalent to the dynamics of the interface model defined in Section 2. Let $\mathsf{S}_i(C)$ denotes the substring corresponding to sites from 1 up to and including i of a configuration C of the qDDE model. And let $h_i(C)$ be the length of the IS corresponding to this substring $\mathsf{S}_i(C)$. It is easy to see that h_i's are non-negative integers and $h_{i+1} - h_i = \pm 1$. The set $\{h_i(C)\}$ will be the same for all configurations C in the same equivalence class $\mathbb{C}$. Hence the set $\{h_i(C)\}$ is a function of only the equivalence class $\mathbb{C}$ and we may write it as $\{h_i(\mathbb{C})\}$. Moreover the correspondence between long-lived equivalence classes and $\{h_i\}$ is one to one in the case of open boundary condition. Hence every long-lived equivalence class can be uniquely represented by a set $\{h_i\}$. We identify h_i's with the height variables of the interface model defined in Section 2. Then, the transitions between these equivalence classes in the qDDE model give rise to a Markovian time-evolution of the interface model.

We now proceed to derive the transition rates for this interface dynamics and show that they correspond to special values of λ_1, λ_2 and λ_3.

As h_i represents the length of the IS of the substring up to site i, h_i changes only when the spins at sites i and $i+1$ in the qDDE model are flipped. These can happen only if they have the same color, in which case $h_{i-1} = h_{i+1}$. Again, as $|h_{i+1} - h_i| = 1$, the only allowed transitions are of the type

$$\{\ldots, h, h-1, h,\ldots\} \longleftrightarrow \{\ldots, h, h+1, h,\ldots\}$$

The rates for this transition will depend on the second neighboring heights h_{i-2} and h_{i+2}. To find the transition rates, let S be the local color configuration at the sites $i-2$ to $i+2$. We write S as $s_1 s_2 s_3 s_4 s_5$, where s_1, s_2, s_3, s_4 and s_5 represents the color at sites $i-2$, $i-1$, i, $i+1$ and $i+2$ respectively. Each s_i can be any of the colors a, b, c, d,... and their different combinations will correspond to different local height configurations $\{h_i\} = \{..., h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2},...\}$. We discuss them one by one.

Case I. $\{h_i\} = \{..., h, h+1, h, h+1, h,...\}$. In this case S is given by *abbcc*. Consider the transition $cc \to bb$. As a result S becomes *abbbb*. As explained earlier, this state is very short lived. There are 3 pairs of b that can flip. It is easy to see that flipping the first or the third *bb* pair does not change h_i. Flipping the middle *bb* pair increases h_i to h_i+2. Since all the 3 *bb* pairs have equal change of flipping, the effective rate for the middle pair to flip is 1/3. There is another possible way for $h_i \to h_i+2$. Starting from *abbcc*, first *bb* flips to *cc* and then the middle *cc* pair flips. Thus the net effective rate for $h_i \to h_i+2$ is 2/3 in this case.

Case II. $\{h_i\} = \{..., h+2, h+1, h, h+1, h,...\}$. In this case S is of the type *abcdd* such that to the left of a there is one of each a, b and c with fully reducible substrings in between them. Consider the transition $dd \to cc$. The new S is *abccc*. The flipping of the first *cc* pair will increase h_i by 2. Whereas flipping of the second *cc* pair will not change h_i. Since both these pairs have equal change of flipping, the rate for $h_i \to h_i+2$ is 1/2 in this case.

Case III. $\{h_i\} = \{..., h, h+1, h, h+1, h+2,...\}$. For these heights S is of the type *abbcd*. The only sequence of transitions which changes h_i is first $bb \to cc$, followed by the flipping of second *cc* pair. As in the previous case the rate for this transition, which increases h_i by 2, is 1/2 for this case.

Case IV. $\{h_i\} = \{..., h+2, h+1, h, h+1, h+2,...\}$. For this case S is of the type *abcde* with no colors equal. Hence the transition rate out of this local configuration is zero.

The remaining four cases corresponds to reverse transitions ($h_i \to h_i-2$) of the above four cases. For example, if the initial heights are $\{h_i\} = \{..., h-2, h-1, h, h-1, h-2,...\}$, h_i can only decrease by 2 and it corresponds to the reverse transition of Case I. Arguing as before, it is easy to determine the rates of these reverse transitions, and they are found to be the same as that of the corresponding $h_i \to h_i+2$ transition. Thus the qDDE model in the large q limit corresponds to the interface model of Section 2 with $\lambda_1 = 2/3$, $\lambda_2 = \lambda_3 = 1/2$.

7. MONTE CARLO AND NUMERICAL DIAGONALIZATION STUDIES OF THE INTERFACE MODEL

We have studied the interface model by using both Monte Carlo simulations and exact numerical diagonalization of the stochastic matrix. Monte Carlo simulations shows that the average height of the interface shows a scaling form

$$\langle h(t, L)\rangle \sim L^{\alpha} f(t/L^{z}) \tag{12}$$

where L is the length of the lattice. The scaling function $f(x) \to x^{\alpha/z}$ as $x \to 0$ and it become a constant in the limit $x \to \infty$. In Section 3 we have shown that $\alpha = 1/2$.

We have done Monte Carlo simulations for various lattice sizes. These are shown in Fig. 5. These can be collapsed into a singe curve using the scaling form (12) with $\alpha = 1/2$ and the dynamical exponent $z = 2.5$. This is shown in Fig. 6.

We have also determined the dynamical exponent z by numerical diagonalization of the stochastic matrix for small systems and extrapolating the results to infinite L. For numerical studies, it is more convenient to work on a ring of size L with periodic boundary conditions.

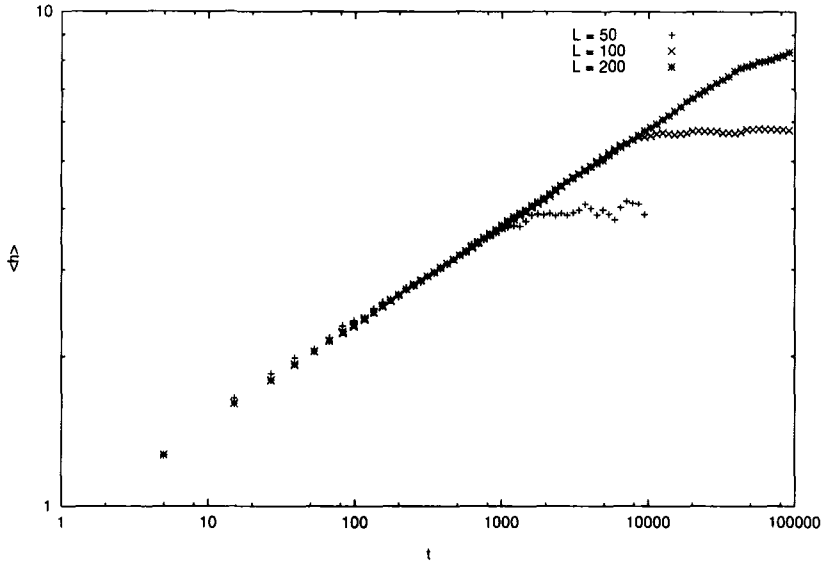


Fig. 5. Average height $\langle h\rangle$ of the interface as a function of time for various lattice sizes.

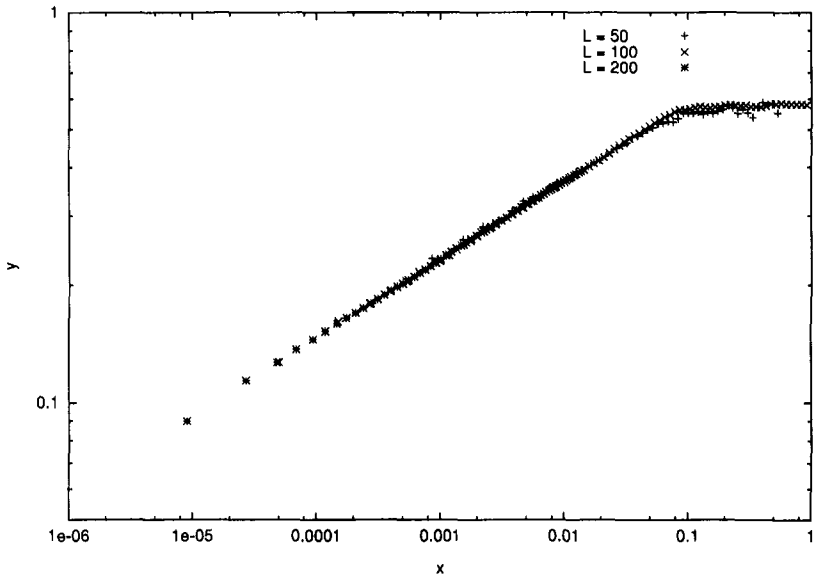


Fig. 6. Collapse of the various curves in Fig. 5. $y = \langle h \rangle L^{-1/2}$ is plotted as a function of the scaling variable $x = t/L^{5/2}$ for various lattice sizes.

However, the use of periodic boundary conditions has a different meaning in the interface model and in the qDDE model. For the qDDE model, periodic boundary conditions mean that if site 1 and site L have the same color, they can flip together to a different color. In terms of the polymer dynamics, this then corresponds to the dynamics of a ring polymer, which diffuses on the Bethe lattice. For the interface model, the periodic boundary conditions are naturally implemented by not holding the end points of chain fixed, but joining them, and let them evolve as elsewhere on the interface. But this is not consistent with the one-to-one correspondence between configurations of the interface model and the long-lived equivalent classes of the qDDE model, which requires that height at origin be always zero. In fact, many distinct configurations of the height model on a ring can correspond to same equivalence class under recoloring of the qDDE model on a ring. For example, for $L = 6$, there are only two distinct equivalence classes of qDDE model (corresponding to $\sqsubset$ or Y shaped ring polymers), but the height model with periodic boundary conditions has 10 configurations. This difference between these models is perhaps responsible for the different behavior of the logarithmic corrections in the two models (see below).

Our procedure of numerical diagonalization is very similar to the one we have used in an earlier study of the TDE model.[12] Here we only briefly out line our procedure. First we reduce the size of the matrix to be diagonalized using symmetries like translation and reflection. Then we find the second largest eigenvalue of the stochastic matrix by numerical diagonalization. Since the largest eigenvalue is zero, this will give the gap Δ in the eigenvalue spectrum. Assuming that the gap scales with system size as $\Delta_L \sim L^{-z}$, we define an effective dynamical exponent,

$$z_L = \frac{\log[\Delta_{L-2}/\Delta_L]}{\log[L/(L-2)]}. \tag{13}$$

The true dynamical exponent z is then obtained by extrapolating z_L to $L = \infty$. We were able to go up to $L = 20$. The size of the matrix to be diagonalized is much smaller than that for the $q = 3$ case of the qDDE model,[13] a simplification achieved in the $q \to \infty$ limit. We diagonalized the stochastic matrix of the interface model for various values of λ_1 keeping $\lambda_2 = \lambda_3 = 1$. The exponent obtained by extrapolating z_L to $L = \infty$ is about 2.8, and it depends on λ_1. In Fig. 7, we have plotted the value of $\Delta(L)$

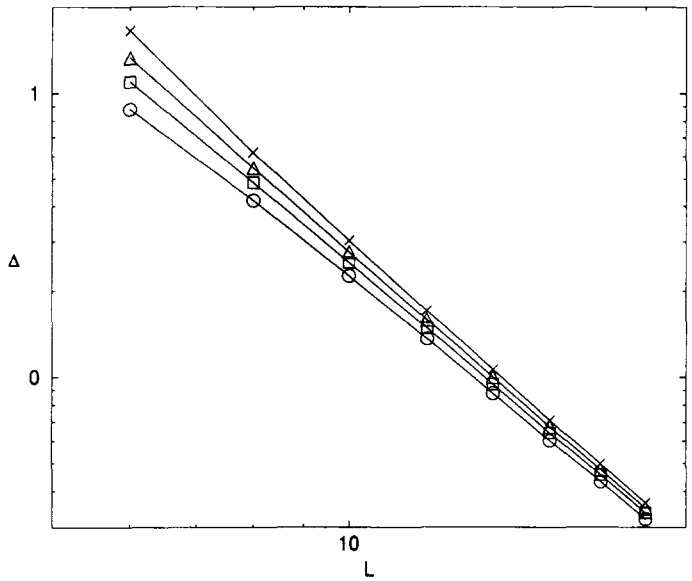


Fig. 7. Plot of Δ vs L on a Log-Log scale for $\lambda_2 = \lambda_3 = 1$ and different values of $\lambda_1 = 0.75$ (○), 1.0 (□), 1.3333 (△), 2 (×). Numerically determined values of the slope of these lines are 2.82, 2.90, 2.96 and 3.05.

versus L on a log-log plot for various values of λ_1. The data do not falls into parallel straight lines. Assuming that this is due to corrections to the scaling of the gap, we tried scaling of the form

$$\Delta(L) = AL^{-z_1} + BL^{-z_2} \tag{14}$$

The best fit were obtained for values of z_1 and z_2 both very close to 2.5. This suggested the extrapolation form

$$\Delta \sim L^{-z}/\log(L/a) \tag{15}$$

In Fig. 8 we have plotted $1/(\Delta L^{2.5})$ vs $\log(L)$ for various values of λ_1, keeping $\lambda_2 = \lambda_3 = 1$. Note that the plot involves no fitting parameters, except for the exponent 2.5. The constant a in equation (15) just gives an overall shift to the plot and hence can be taken to be 1. We find that all points falls into straight lines, which gives us some confidence that equation (15) is likely to be the correct scaling form.

To see if the logarithmic correction term is responsible for the poor convergence of observed effective dynamical exponent in the earlier exact diagonalization study of the TDE and 3-color qDDE models,[12, 13] we

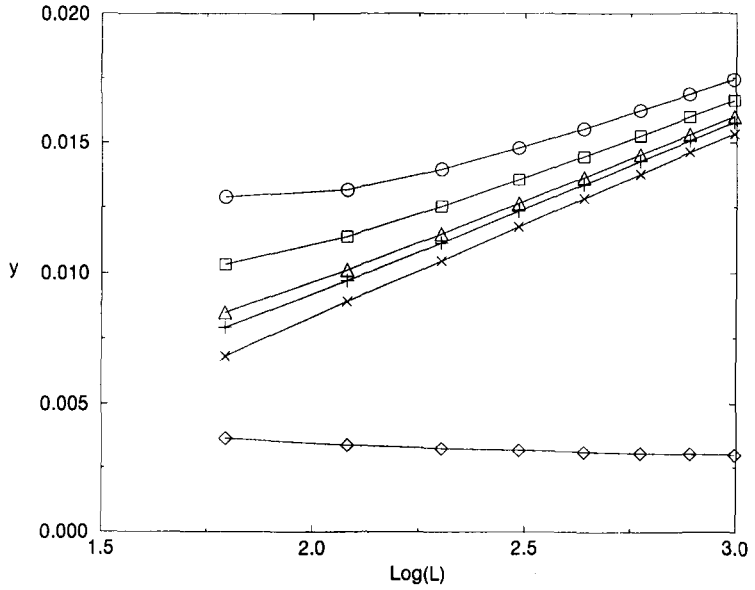


Fig. 8. Plot of $y = 1/(\Delta L^{2.5})$ vs $\log(L)$ for A) $\lambda_2 = \lambda_3 = 1$ and different values of $\lambda_1 = 0.75$ (○), 1.0 (□), 1.3333 (△), 1.5 (+), 2 (×), B) for the 3-color qDDE model (◇).

have also plotted earlier data for the 3-color qDDE model in Fig. 8. The graph is again fairly linear with very small *negative* slope. As a negative slope is not possible asymptotically, we conclude that the 3-DDE model shows no evidence of a logarithmic correction term.

We do not understand fully the reason for the different scaling of gap with L in the qDDE model and the interface model for periodic boundary conditions. Certainly, it comes from the already-mentioned fact that the periodic boundary conditions in the two cases mean different things. To be more specific, making the boundary conditions periodic in the interface model involves enlarging the space of configurations to include those configurations where the height at $x=0$ is not zero. [It is easy to see that in this case the total number of allowed height configurations is approximately L times the number of different equivalence classes of the qDDE model.] Thus, the relaxation matrix is much larger for the height model than for the qDDE model, and there are many extra modes. The eigenvalues of some of these extra modes fall in the gap of the qDDE model. We emphasize that this phenomena is boundary-condition dependent. For open boundary conditions, we have shown that the two models are exactly equivalent in the large q limit. All bulk correlation functions which are independent of boundary conditions will behave the same way in the two models (and presumably not show any log-corrections to scaling).

8. CONCLUDING REMARKS

It is of some interest to study this model when the rate $h \to h+2$ is different from the rate of the reverse transition. In the particle language, this corresponds to making the rates of the leftward hops of particles different from the rightward ones. When λ_4 is non-zero, this changes the dynamical exponent from the Hammersley–Edwards–Wilkinson value 2 to the KPZ value 3/2. When λ_4 is zero, the effect of adding asymmetry is much stronger. In this case, it is easy to see that we can define a Hamiltonian function $\mathscr{H} = -A \sum_i h_i$, such that the transition rates satisfy detailed balance with respect to this $\mathscr{H}$. Here A is a constant, which depends on the microscopic rates. The probability of a configuration in the steady state is then given by the Boltzmann factor $\exp(-\mathscr{H})$. Let r be the ratio of the rates for $h \to h+2$ and $h \to h-2$. When $r \ll 1$, starting from a configuration where the minimum height is zero, the interface prefer to move downwards. Because of the conservation of minimum height, at any point the height cannot decrease for ever. In this case the steady state would consists of the profile $h_i = 0$ (1) for i even (odd) and small fluctuations about this (or $h_i = 1$ (0) for i even (odd), depending up on the initial profile). In the particle language this corresponds to an "antiferromagnetic" type of

order (0101...). This interface is similar to an interface which is adsorbed to a wall. When $r=1$ we know that the interface is not bound to the wall. Hence there is presumably a phase transition from a wet (bound) phase to a non-wet phase as r increases from 0 to 1. We have not yet studied the model in detail to find the nature of this transition. Recently Hinrichsen *et al.* has studied a non-equilibrium SOS interface model in the presence of a wall that shows a similar wetting transition.[26] When the rate $r \gg 1$, the steady state consists of small fluctuations about a single mountain profile with maximum height approximately equal to the maximum possible value $L/2$. In particle language this corresponds to a phase separation with all particles coming together in one half of the lattice. If we start with a random configuration of particles and holes, such a system will coarsen with time. But the coalescence process needs crossing free-energy barriers. In order to merge two islands of size l, needs an activation energy of order l. As a result the average domain size grows only logarithmically with time. Again, a detailed study has not done so far. Similar models, showing phase separation in one-dimensional systems with local evolution rules, have recently been studied by other authors. Lahiri and Ramaswami[27] studied a 1-dimensional lattice model of sedimenting colloids, which also shows logarithmic coarsening with time. Evans *et al.*[28] have rigorously proved the existence of phase separation in a driven diffusive model involving 3 species of particles.

It is easy to construct higher dimensional interface growth models having the property that minimum of heights is conserved locally. These are however may not realizable as the $q \to \infty$ limit of higher dimensional qDDE models. Consider an interface model on a square lattice where the heights $h(i, j)$ are integers and nearest neighbor slopes takes values ± 1. The growth rule is that $h(i, j) \to h(i, j) + 2$, if all neighbors have height $h(i, j) + 1$ and at least one of the 4 second neighbors $((i \pm 2, j), (i, j \pm 2))$ has height $h(i, j)$. Reverse transition takes place with the same rate. In this model also $\mathrm{Min}(\{h(i, j)\})$ is conserved. Mountain craters analogue of that in Fig. 2 are easy to construct such that it takes a long process of restructuring to go from one to another.

Another simple variant of this model which avoids the odd-even sublattices is the following: We consider an RSOS model with an integer height coordinate $h(i, j)$ at the site (i, j) of a square lattice. The difference in heights at two adjacent sites is constrained to be one of three values: -1, 0 or 1. Thus we allow neighboring sites to have the same height. In the initial configuration, $h(i, j) = 0$ for all sites (i, j). The transition rule is that $h(i, j) \to h(i, j) + 1$ with rate 1, if this would not violate the RSOS constraint in the final state, and if at least one of the neighbors have the same height as (i, j) in the initial state. The reverse transition also occurs at the

same rate. Clearly this also gives rise to a fluctuating surface whose minimum height value does not change in time. A detailed investigation has not yet been undertaken.

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